Chapter 1

Introduction

In the classical theory of thermodynamics, heat conduction is viewed as a purely diffusive Process, typically described using Fourier's Law. As a result, we get the usual heat equation. This equation provides a useful description of heat conduction under a large range of conditions and predicts an infinite speed of propagation, that is any thermal disturbance at one point has an instantaneous effect elsewhere in the body. Experiments have shown showed that heat conduction in some dielectric crystals at low temperatures is free of this paradox (infinite speed propagation) and disturbances which are almost entirely thermal, may propagate with a finite speed. This wave-like behavior of heat propagation and is known as second sound. It was first detected in Helium, He, and then in high purity dielectric crystals of sodium floride, NaFl, and bismath Bi. The range of temperature, for which the second sound is detectable, is in fact quite small and normal diffusive propagation takes place above it.

In this theory [2], [3] it is assumed that the heat flux satisfies Cattaneo's law and, thus, we obtain a hyperbolic system which describes propagation of the heat. Another approach to the second sound is by introducing an internal parameter, which accounts for a history memory effect of heat flux [6], [19], [20]. The effect of memory may be considered as a functional of the history of temperature gradient. In fact, this is the approach, which we follow in this thesis. This work is divided into four chapters, the first chapter is introduction and in the second chapter we review the basic results of partial differential equations and put more stress on the wave equation, characteristic equations and systems of first order linear equations. We also review the classifications of second-order equations.

In chapter three, we investigate the literature dealing with the breakdown of classical solutions of nonlinear hyperbolic systems and present some pertinent results. We also introduce the classical theory of heat conduction.

In the last chapter, we consider our problem and establish our main result dealing with the formation of singularities in the classical solution of a hyperbolic system describing the propagation of heat by second sound.

Chapter 2

First and Second Order Partial Differential Equations

2.1 Preliminary Notation and Concepts

Definition 2.1.1 The order of a partial differential equation is order of the highest partial derivative appearing in the equation. The partial derivatives of independent variable are $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{dx^2}, \frac{\partial^2 u}{\partial y \, \partial x}$ and $\frac{\partial^2 u}{\partial y^2}$ are sometimes denoted by u_x, u_y, u_{xx}, u_{xy} and u_{yy} (or p, q, r, s, and t), respectively. The most general first-order partial differential equation with two independent variables x and y has the form

$$F(x, y, u, p, q) = 0, \quad p = u_x, \quad q = u_y. \tag{2.1.1}$$

The most general second-order partial differential equation is of the form

$$F(x, y, u, p, q, r, s, t) = 0, \quad r = u_{xx}, \quad s = u_{xy}, \quad t = u_{yy}. \tag{2.1.2}$$

Definition 2.1.2 A partial differential equation is said to be linear if the unknown function u and all its partial derivatives appear in an algebraically linear form, i.e., as an expression of the first degree. For example, the equation

$$a_{11}u_{xx} + 2a_{12}\,u_{xy} + a_{22}\,u_{yy} + b_1 u_x + b_2 u_y + c_0 u = f, \tag{2.1.3}$$

where the coefficients $a_{11}, a_{12}, a_{22}, b_1, b_2,$ and c_0 and f are functions of x and y, is a second-order linear partial differential equation in the unknowns x, y and u.

Definition 2.1.3 A differential operator L is linear if $L(\alpha u + \beta v) = \alpha Lu + \beta Lv$ where α and β are scalars, and u and v are any functions with continuous partial derivatives of appropriate order.

Definition 2.1.4 A partial differential equation with L linear is said to be homogeneous if $Lu = 0$ whereas the equation $Lu = g$, where $g \neq 0$ is a given function of the independent variables, is said to be nonhomogeneous.

Definition 2.1.5 A function ϕ is said to be a solution of the partial differential equation if ϕ and its partial derivatives, when substituted for u and its partial derivatives occurring in the partial differential equation, reduce it to an identity in the independent variables.

Definition 2.1.6 The general solution of a linear partial differential equation is a linear combination of all linearly independent solutions of the equation with as many arbitrary functions as the order of the equation.

Definition 2.1.7 A partial differential equation is said to be quasi-linear if it is linear in all the highest-order derivatives of the dependent variable. For example, the most general form of a quasi-linear second-order equation is

$$A(x,y,u,p,q)u_{xx}+B(x,y,u,p,q)u_{xy}+C(x,y,u,p,q)U_{yy}+f(x,y,u,p,q)=0.$$

2.2 Characteristic Equations for the First Order Linear Equation

Consider the equation

$$a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}+c(x,y)u=f(x,y) \tag{2.2.1}$$

where a, b, c and f are functions of $(x, y) \in D \subset \mathbb{R}^2$, with $ab \neq 0$. If $f = 0$, the PDE is said to be homogeneous. Now we introduce a transform (change of variables), $\xi = \phi(x, y)$ and $\eta = \psi(x, y)$ such that (2.2.1) takes a simpler form

$$w_\eta(\xi,\eta)+h(\xi,\eta)w=F(\xi,\eta). \qquad (2.2.2).$$

Equation (2.2.2) behaves like an ordinary differential equations. This change of variables must be one-to-one so that (x, y) can be expressed in terms of (ξ, η). This, of course, implies that $J = \begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix} \neq 0$, at least in a domain D.

The partial derivatives become

$$u_x = u_\xi\, \xi_x + u_\eta\, \eta_x, \quad u_y = u_\xi\, \xi_y + u_\eta\, \eta_y$$

so that (2.2.1) gives

$$(a\xi_x + b\xi_y)\, u_\xi + (a\eta_x + b\eta_y)\, u_\eta + cu = F(\xi, \eta).$$

We get $a\eta_x + b\eta_y = 0$, this implies that $\frac{\eta_x}{\eta_y} = -\frac{b}{a}$, if $\eta_y \neq 0$ in domain D.

If such η exists then the curve $\eta(x, y) = c$ satisfies $d\eta = \eta_x\, dx + \eta_y\, dy = 0$, this implies that $\frac{dy}{dx} = \frac{-\eta_x}{\eta_y} = \frac{b}{a}$.

Therefore, $\eta = \psi(x, y)$ is an integral to the ordinary differential equation $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$.

Definition 2.2.1 Given the first order PDE (2.2.1), we call the equation $\dfrac{dy}{dx} = \dfrac{b(x, y)}{a(x, y)}$, the characteristic equation of (2.2.1)

The Cauchy Problem

Definition 2.2.2 The equation (2.2.1) associated with u prescribed function on a curve Γ, is called a Cauchy problem.

Theorem 2.2.1 [8], [22] Given the Cauchy problem

$$\begin{cases} a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y) \\ u|_\Gamma = h \end{cases}$$

1. The Cauchy problem has a unique solution for any $h \in C^1$ if Γ is not a characteristic curve.

2. If Γ is a characteristic curve then the problem has either no solution or infinite many solutions.

2.3 Classification of Second-Order Equations

The most general form of a second-order homogeneous linear equation is

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0. \tag{2.3.1}$$

To show a correspondence of this equation with algebraic quadratic equation, we replace u_x by α, u_y by β, u_{xx} by α^2, u_{xy} by $\alpha\beta$, and u_{yy} by β^2 then the left side of (2.3.1) reduces to a second-degree polynomial in α and β :

$$P(\alpha, \beta) = A\alpha^2 + B\alpha\beta + C\beta^2 + D\alpha + E\beta + F. \tag{2.3.2}$$

It is known from analytical geometry and algebra that the polynomial equation $P(\alpha, \beta) = 0$ represents a hyperbola, parabola, or ellipse according to its discriminant $B^2 - 4AC$ is positive, zero, or negative. Thus, equation (2.3.1) is classified as hyperbolic, parabolic, or elliptic according as the quantity $B^2 - 4AC$ is positive, zero, or negative.

2.4 Canonical Form for Hyperbolic Equations

Consider the most general transformation of the independent variables x and y of the equation

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{2.4.1}$$

to new variables

$$\xi, \eta, \text{ where } \xi = \xi(x, y), \quad \eta = \eta(x, y) \tag{2.4.2}$$

such that the functions ξ and η are continuously differentiable and the Jacobian

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = (\xi_x \eta_y - \xi_y \eta_x) \neq 0 \tag{2.4.3}$$

in the domain Ω, where the Eq. (2.4.1) holds. Using the chain rule, the partial derivatives become

$$\begin{aligned} u_x &= u_\xi \, \xi_x + u_\eta \eta_x \\ u_y &= u_\xi \, \xi_y + u_\eta \, \eta_y \\ u_{xx} &= u_{\xi y} \, \xi_x^2 + 2u_{\xi\eta} \;\; \xi_x \eta_x + u_{\xi\eta} \, \eta_x^2 + u_\xi \, \xi_{xx} + u_\eta \, \eta_{xx} \\ u_{xy} &= u_{\xi\xi} \, \xi_x \, \xi_y + u_{\xi\eta} \, (\xi_x \eta_y + \xi_y \, \eta_x) + u_{\eta\eta} \, \eta_x \eta_y + u_\xi \, \xi_{xy} + u_\eta \, \eta_{xy} \\ y_{yy} &= u_{\xi\xi} \, \xi_y^2 + 2u_{\xi\eta} \, \xi_y \, \eta_y + u_{\eta\eta} \, \eta_y^2 + u_\xi \, \xi_{yy} + u_\eta \, \eta_{yy}. \end{aligned} \tag{2.4.4}$$

Substituting these expressions into the original differential equation (2.4.1), we get

$$\overline{A} \, u_{\xi\xi} + \overline{B} \, u_{\xi\eta} + \overline{C} \, u_{\eta\eta} + \overline{D} u_\xi + \overline{E} u_\eta + \overline{F} u = \overline{G} \tag{2.4.5}$$

where

$$
\begin{aligned}
\overline{A} &= A\,\xi_x^2 + B\,\xi_x\,\xi_y + C\xi_y^2 \\
\overline{B} &= 2A\,\xi_x\,\eta_x + B\,(\xi_x\,\eta_y + \xi_y\eta_x) + 2C\xi_y\,\eta_y \\
\overline{C} &= A\eta_x^2 + B\eta_x\,\eta_y + C\eta_y^2 \\
\overline{D} &= A\,\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
\overline{E} &= A\,\eta_{xx} + B\,\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
\overline{F} &= F, \quad \overline{G} = G
\end{aligned}
\tag{2.4.6}
$$

Since the classification of Eq. 2.4.1 depends on the coefficients A, B and C, we can also rewrite the equation in the form

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} = H(x, y, u, u_x, u_y) \tag{2.4.7}$$

It can be shown easily that under the transformation (2.4.2), Eq. (2.4.7) takes one of the following three canonical forms:

$$
\begin{aligned}
(i)\, u_{\xi\xi} - u_{\eta\eta} &= \phi(\xi, \eta, u, u_\eta, u_\eta) \\
\text{or } u_{\xi\eta} &= \phi(\xi, \eta, u, u_\xi, u_\eta) \quad \text{in the hyperbolic case} && (2.4.8) \\
(ii)\, u_{\xi\xi} + u_{\eta\eta} &= \phi(\xi, n, u, u_\xi, u_\eta) \quad \text{in the elliptic case} && (2.4.9) \\
(iii) \qquad u_{\xi\xi} &= \phi(\xi, \eta, u, u_\xi, u_\eta) && (2.4.10) \\
\text{or} \qquad u_{\eta\eta} &= \phi(\xi, \eta, u, u_\xi, u_\eta) \quad \text{in the parabolic case .}
\end{aligned}
$$

Since the discriminant $B^2 - 4AC > 0$ for hyperbolic case, we set $\overline{A} = 0$ and $\overline{C} = 0$ in Eq. (2.5.6), which will give us the coordinates ξ and η that reduce the given PDE to a canonical form in which the coefficients of $u_{\xi\xi}, u_{\eta\eta}$ are zero. Thus we have

$$\begin{aligned} \overline{A} &= A\xi_x^2 + B\xi_x\,\xi_y + C\xi_y^2 = 0 \\ \overline{C} &= A\eta_x^2 + B\eta_x\,\eta_y + C\eta_y^2 = 0 \end{aligned}$$

which, on rewriting, become

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0 \tag{2.4.11}$$

$$A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0 \tag{2.4.12}$$

Solving these equations for (ξ_x/ξ_y) and (η_x/η_y), we get

$$\frac{\xi_x}{\xi_y} = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \tag{2.4.13}$$

$$\frac{\eta_x}{\eta_y} = \frac{-B - \sqrt{B^2 - 4AC}}{2A} \tag{2.4.14}$$

The condition $B^2 > 4AC$ implies that the slopes of the curves $\xi(x, y) = C_1$, $\eta(x, y) = C_2$ are real. Thus, if $B^2 > 4AC$, then at any point (x, y), there exist two real directions given by the two roots (2.4.13) & (2.4.14) along which the PDE (2.4.1) reduces to the canonical form. These are called characteristic equations. Though there are two solutions for each quadratic, we have considered only one solution for each. Since we end up with the same two coordinates.

Along the curve $\xi(x,y) = C_1$, we have

$$d\xi = \xi_x\, dx + \xi_y\, dy = 0$$

Hence,

$$\frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) \tag{2.4.15}$$

Similarly, along the curve $\eta(x,y) = C_2$, we have

$$\frac{dy}{dx} = -\left(\frac{\eta_x}{\eta_y}\right) \tag{2.4.16}$$

Integrating Eqs. (2.4.15) and (2.4.16), we obtain the equations of family of characteristics $\xi(x,y) = C_1$ and $\eta(x,y) = C_2$ which are called the characteristics of the PDE (2.4.1). To obtain the canonical form for the given PDE, we substitute the expressions of ξ and η into (2.4.5) which reduces it to Eq. (2.4.8).

Example: Let us consider the following example:

$$3u_{xx} + 10u_{xy} + 3u_{yy} = 0$$

Comparing with the standard PDE (2.4.1), we have $A = 3, B = 10, C = 3, B^2 - 4AC = 64 > 0$. Hence the given equation is a hyperbolic PDE. The corresponding characteristics are:

$$\begin{aligned}
\frac{dy}{dx} &= -\left(\frac{\xi_x}{\xi_y}\right) = -\left(\frac{-B+\sqrt{B^2-4AC}}{2A}\right) = \frac{1}{3} \\
\frac{dy}{dx} &= -\left(\frac{\eta_x}{\eta_y}\right) = -\left(\frac{-B-\sqrt{B^2-4AC}}{2A}\right) = 3
\end{aligned}$$

To find ξ and η, we first solve for y by integrating the above equations. Thus we get $y = 3x + c_1, \quad y = \frac{1}{3}x + c_2$ which give the constants as

$$c_1 = y - 3x, \qquad c_2 = u - \frac{x}{3}$$

Therefore

$$\xi = y - 3x = c_1, \quad \eta = y - \frac{1}{3}x = c_2$$

These are characteristic lines for the given hyperbolic equation.

In this example, the characteristics are found to be straight lines in the (x, y) -plane along with the initial data impulses will propagate. To find the canonical equation, we substitute the expressions for ξ and η into eq. (2.4.6) to get

$$\begin{aligned}
\bar{A} &= A\xi_x^2 + B\xi_x\xi_y + c\xi_y^2 = 3(-3)^2 + 10(-3)(1) + 3 = 0 \\
\bar{B} &= 2A\xi_x\eta + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2c\xi_y\eta_y \\
&= 2(3)(-3)\left(-\frac{1}{3}\right) + 10\left[(-3)(1) + 1\left(\frac{-1}{3}\right)\right] + 2(3)(1)(1) = \frac{-64}{3} \\
\bar{C} &= 0, \ \ \bar{D} = 0, \ \ \bar{E} = 0, \ \ \bar{F} = 0
\end{aligned}$$

Hence, the required canonical form is

$$\frac{64}{3}u_{\xi\eta} = 0 \ \text{ or } \ u_{\xi\eta} = 0$$

On integration, we obtain $u(\xi, \eta) = f(\xi) + g(\eta)$ where f and g are arbitrary. Going back to the original variables, the general solution is $u(x, y) = f(y - 3x) + g\left(y - \frac{x}{3}\right)$.

2.5 Initial and Boundary Conditions

A partial differential equation subject to certain conditions in the form of initial or boundary conditions is known as an initial value or a boundary value problem. The initial conditions, also known as Cauchy conditions, are the values of the unknown function u and an appropriate number of its derivatives at the initial point.

The boundary conditions fall into the following three categories:

(i) Dirichlet boundary conditions, when the values of the unknown function u are prescribed at each point of the boundary ∂D of a given domain D.

(ii) Neumann boundary conditions, when the values of the normal derivative of the unknown function u are prescribed at each point of the boundary ∂D.

(iii) Robin boundary conditions, when the values of a linear combination of the unknown function u and its normal derivtive are prescribed at each point of the boundary ∂D.

2.6 Systems of First Order Linear Equations

Systems of first order partial differential equations arise in many areas of mathematics, engineering, and physical sciences. In this section we will limit our study of systems to those which consist of m equations in m unknowns and only two independent variables, and which are of the form

$$\frac{\partial u_i}{\partial t} + a_{i1}\frac{\partial u_1}{\partial x} + a_{i2}\frac{\partial u_2}{\partial x} + \ldots + a_{im}\frac{\partial u_m}{\partial x} + b_i = 0, \;\; i = 1, 2, \ldots m \qquad (2.6.1)$$

Each of the unknowns $u_i = u_i(x,t)$, $i = 1, 2, \ldots, m$, is assumed to be a function of the two independent variables x and t, while the coefficients a_{ij}, b_i may be functions of the unknowns as well as of x and t, $a_{ij} = a_{ij}(x, t, u_1, \ldots, u_m)$, $b_i = b_i(x, t, u_1, \ldots, u_m)$, $i, j = 1, 2, \ldots, m$.

The classification of the first order system (2.6.1) according to linearity is similar to the classification of single first order equations. If the coefficients a_{ij} depend actually on the unknowns $u_1, \ldots, u_m$, the system is called quasi-linear. The system (2.6.1) can be written in more convenient and compact form by introducing matrix notation.

Let u and b denote the column vectors

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \text{ and } A \text{ be } m \times m \text{ matrix}$$

$$A = [a_{ij}]_{i,j=1,\ldots,m} = \begin{bmatrix} a_{11} & \ldots & a_{1m} \\ \vdots & & \vdots \\ a_{m1} & & a_{mm} \end{bmatrix}$$

Now, it is easy to see that the system (2.6.1) can be written in the form

$$\frac{\partial u}{\partial t} + A(x,t,u)\frac{\partial u}{\partial x} + b(x,t,u) = 0 \qquad (2.6.2)$$

Definition 2.6.1. A solution of the system (2.6.2) in a domain Ω of R^2 is a "vector" function $u = u(x,t)$ which is defined, of class C^1 in Ω and is such that the following two conditions are satisfied:

(i) For every $(x,t) \in \Omega$, the point $(x,t,u(x,t))$ is in the domain of A and b.

(ii) When $u = u(x,t)$ is substituted into (2.6.2), the resulting "vector" equation is an identity in (x,t) for all (x,t) in Ω.

2.7 Linear Hyperbolic Systems and Reduction to Canonical Form

We study in this section linear system of m first order equations in m unknowns and two independent variables of the form

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{m} a_{ij}(x,t)\frac{\partial u_j}{\partial x} + b_i(x,t,u_1,\dots,u_m) = 0, \quad i = 1,2,\dots,m \tag{2.7.1}$$

In the matrix notation the system (2.7.1) has the form

$$\frac{\partial u}{\partial t} + A(x,t)\frac{\partial u}{\partial x} + b(x,t,u) = 0. \tag{2.7.2}$$

We assume that the coefficient matrix $A(x,t)$ is of class C^1 in the domain of the (x,t)-plane under consideration.

Just as in the case of a single partial differential equation it turns out that most of the important properties of the system (2.7.2) depend only on its principal part $u_t + A(x,t)u_x$. Since the principal part is completely characterized by the coefficient matrix $A(x,t)$, this matrix plays a fundamental role in the study of (2.7.2). There are two important classes of system of the form (2.7.2) which are defined in terms of properties of the matrix $A(x,t)$. Recall that an eigenvalue of A is a root $\lambda = \lambda(x,t)$ of the characteristic equation,

$$\det|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22}-\lambda & \dots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm}-\lambda \end{vmatrix} = 0 \tag{2.7.3}$$

and that for each eigenvalue λ of A, there is at least one nontrivial column vector $p = p(x,t) \neq 0$ such that

$$Ap = \lambda p. \tag{2.7.4}$$

As defined to this point, eigenvector are right eigenvectors in the sense that they appears columns on the right side of $n \times n$ matrix A in the equation

$$Ax_i = \lambda_i x_i$$

it is also possible to consider left eigenvector which are multiplied as rows on the left side of A in the form

$$Y_i^\top A = \lambda Y_i^\top$$

The equation can be rewritten in column form by taking the transpose, yielding

$$A^\top y_i = \lambda_i Y_i.$$

Therefore, a left eigenvector of A is the same thing as an ordinary right eigenvector of $A^\top$.

Definition 2.7.1. The system (2.7.2) is said to be strictly hyperbolic at the point (x, t), if $A(x, t)$ has m real and distinct eigenvalues

$$\lambda_1(x,t) < \lambda_2(x,t) < \ldots, < \lambda_m(x,t),$$

and it is said to be strictly hyperbolic in a domain of R^2 if it is strictly hyperbolic at every point of the domain.

Since the eigenvalues $\lambda_k(x, t), k = 1, 2, \ldots, m$ are assumed to be distinct, it follows from a well-known theorem of linear algebra that the corresponding eigenvectors

$$P_k(x,t) = \begin{bmatrix} P_{1k}(x,t) \\ P_{2k}(x,t) \\ \vdots \\ P_{mk}(x,t) \end{bmatrix}; \; k = 1, 2, \ldots, m \tag{2.7.5}$$

are linearly independent. Remember that the eigenvector P_k corresponding to the eigenvalue λ_k is determined only up to a multiplicative constant. Any multiple of P_k is also an eigenvector corresponding to λ_k.

We will now show that if the system (2.7.2) is strictly hyperbolic at point (x, t), it is possible to obtain a very simple canonical form of the system in a neighborhood of (x, t) by introducing new unknowns.

Let Λ be the $m \times m$ diagonal matrix with diagonal entries the eigenvalues of A, and let p be the $m \times m$ matrix with columns the corresponding eigenvectors (2.7.5) of A,

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & & & \lambda_m \end{bmatrix}, \quad P = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1m} \\ p_{21} & p_{22} & \dots & p_{2m} \\ \vdots & \vdots & & \vdots \\ p_{m1} & p_{m2} & & p_{mm} \end{bmatrix}. \tag{2.7.6}$$

Since the eigenvalues of A are distinct and continuous at (x, t), they remain distinct in some neighborhood U of (x, t), and the columns of P are linearly independent in U. From linear algebra [1], P is nonsingular in U and if P^{-1} denotes its inverse, then

$$P^{-1}AP = \Lambda \text{ in } U. \tag{2.7.7}$$

We introduce now the new unknown v by the relation

$$v = P^{-1}u \tag{2.7.8}$$

Then

$$u = Pv \tag{2.7.9}$$

and

$$\frac{\partial u}{\partial t} = P\frac{\partial v}{\partial t} + \frac{\partial P}{\partial t}v, \quad \frac{\partial u}{\partial x} = P\frac{\partial v}{\partial x} + \frac{\partial P}{\partial x}v \tag{2.7.10}$$

Substituting (2.7.9) and (2.7.10) into (2.7.2) we obtain

$$P\frac{\partial v}{\partial t} + AP\frac{\partial v}{\partial x} + \frac{\partial P}{\partial t}v + A\frac{\partial P}{\partial x}v + b(x, t, Pv) = 0. \tag{2.7.11}$$

Finally, multiplying this equation from the left by P^{-1} and using (2.7.7) we get the desired canonical form in the neighborhood U of (x, t),

$$\frac{\partial v}{\partial t} + \Lambda(x,t)\frac{\partial v}{\partial x} + c(x,t,v) = 0 \tag{2.7.12}$$

$$c = P^{-1}\left(\frac{\partial P}{\partial t} + A\frac{\partial P}{\partial x}\right)v + P^{-1}\,b(x,t,P,v).$$

The simplicity of the canonical form (2.7.12) becomes apparent if we write it in the form,

$$\frac{\partial v_i}{\partial t} + \lambda_i(x,t)\frac{\partial v_i}{\partial x} + c_i(x,t,v_1,v_2,\dots,v_m) = 0 \quad i = 1,2,\dots,m. \tag{2.7.13}$$

The principal part of the ith equation involves only the ith unknown v_i.

2.8 Riemann Invariants

Definition 2.8.1 A k-Riemann invariant is a function $r_k : R^2 \to R$ satisfying

$$\nabla r_k \cdot \vec{V}_k = 0 \tag{2.8.1}$$

From (2.8.1), r_k is constant along the trajectories of the vector field $\vec{V}_k$, so it is elementary to find Riemann invariants. In fact, we can choose, r_1 and r_2 to have nondegenerate dependence on u_1 and u_2. Indeed, to find the function r_1, simply take any curve C which is transverse to the trajectories of $\vec{v}_1$, and assign arbitrary, but strictly increasing, values for r_1 along C. In this way, each trajectory of $\vec{v}_1$ is given a distinct constant value for r_1. Similarly, the function r_2 may be defined so that it has distinct constant value along each trajectory of $\vec{v}_2$. One may use the Riemann invariants r_1, r_2 in place of u_1, u_2 as new dependent variables in the hyperbolic system.

To simplify the notation, let $r = r_1$ and $s = r_2$, similarly, let $\lambda = \lambda_1$ and $\mu = \lambda_2$. If $\vec{u}$ is a solution of

$$\vec{u}_t + A(\vec{u})u_x = 0 \tag{2.8.2}$$

consider $r(x,t) = r(\vec{u}(x,t))$. Then

$$r_t + \mu\, r_x = \nabla_r \cdot \vec{u}_t + \mu\, \nabla_r \cdot \vec{u}_x = \nabla_r \cdot (-A(\vec{u}) + \mu)\vec{u}_x = 0$$

where we have used the fact that ∇_r must be a left eigenvector for $A(\vec{u})$ with eigenvalue μ; notice that $r_t + \mu\, r_x = 0$, simply state that r is constant along the characteristic $\frac{dx}{dt} = \mu$. Similarly, $s_t + \lambda\, s_x = 0$, i.e. S is constant along characteristic $\frac{dx}{dt} = \lambda$. This means that when expressed in terms of these new variables, the system (1.8.2) is diagonalized:

$$\vec{v}_t + \Lambda \vec{v}_x = 0 \text{ where } \overline{v} = \begin{pmatrix} r \\ s \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix} \tag{2.8.3}$$

One application of (2.8.3) is to determine when a solution develops singularities.

Chapter 3

The Classical Solutions of quasilinear Hyperbolic Systems

3.1 Preliminary Lemmas about Ordinary Differential Equation

Generally, classical solutions of a quasilinear hyperbolic system develop singularities in finite time and the strategy used to prove that depends on the following two simple theorems:

Theorem 3.1.1 [5]

Let $z(t)$ be the solution of the initial-value problem

$$\frac{dz}{dt} = a(t)z^2, \quad z(0) = m \tag{3.1.1}$$

in the interval $(0, T)$. Suppose that the function $a(t)$ satisfies the inequality $0 < A \leq a(t), \quad 0 \leq t \leq T$ and suppose that m is positive; then $T \leq (mA)^{-1}$.

Theorem 3.1.2 [4]

Suppose that $a(t)$ satisfies the inequality $|a(t)| < B$; then the initial value problem (3.1.1) has a solution for $|t| < |Bm|^{-1}$.

By using theorem (3.1.1) and (3.1.2) we can place the upper and lower bounds on the time interval in which the solution of a given initial-value problem exists.

In a similar way the quasilinear hyperbolic system of first order partial differential equations in the two variables has been studied with additional structure provided by certain quantities, called Riemann invariants, which are constant along the characteristics.

3.2 Formation of Singularities in the One-Dimensional Cauchy Problem

3.2.1 Nonlinear Elasticity.

In the absence of external forces, the equations governing the motion of homogeneous one-dimensional body, with unit reference density, take the form

$$\begin{cases} \epsilon_t(x,t) = v_x(x,t) \\ v_t(x,t) = \sigma_x(x,t) \end{cases} \tag{3.2.1}$$

where ϵ is the strain, v is the velocity, and σ is the stress. For an elastic body the stress is determined by the strain through the constitutive relation

$$\sigma(x,t) = p(\epsilon(x,t)) \tag{3.2.2}$$

we assume that

$$p'(\xi) > 0 \qquad -1 < \xi < \infty, \tag{3.2.3}$$

and consequently the system (3.2.1) is strictly hyperbolic.

To see the destabilizing effect of the nonlinear elastic response we study the evolution of the strain ϵ and the velocity v, as well as their first partial derivatives, along the characteristic curves described by

$$\frac{dx}{dt} = \pm\,[p'(\epsilon)]^{\frac{1}{2}}. \tag{3.2.4}$$

We assume, in addition to (3.2.3) that

$$p''(0) > 0. \tag{3.2.5}$$

Solutions to the equation (3.2.1), with the initial conditions,

$$\epsilon(x,0) = \epsilon_0(x), \qquad v(x,0) = v_0(x), \tag{3.2.6}$$

generally develop singularities in finite time. This was established by Lax [5] in 1964 and MacCamy and Mizel [7] in 1967. In his work, Lax studied the evolution of certain quantities along the characteristics and showed, under hypotheses (3.2.3), (3.2.5) on p, that smooth solutions generally develop singularities in finite time no matter how smooth and small the initial data are. MacCamy and Mizel [7] allowed p'' to change sign. They also showed, under appropriate conditions on P, that there can exist intervals of x in which the solution must exist for all time t even though it breaks down for some x-values outside these intervals.

Proposition 3.2.1 Assume that (3.2.3), (3.2.5) hold. Then there are $\epsilon_0, v_0 \in C_0^\infty(\mathbb{R})$ for which the solution of (3.2.1), (3.2.6) develops singularities in finite time.

This proposition is due to Lax [5].

Also, for a strictly hyperbolic system of the form:

$$\begin{cases} u_t(x,t) = a(u(x,t),\, v(x,t))\, v_x(x,t) \\ v_t(x,t) = b(u(x,t),\, v(x,t))\, u_x(x,t), \end{cases} \tag{3.2.7}$$

Messaoudi [13] showed that, for $a, b \,\epsilon\, C^2$ strictly positive functions, the same result in [4], [6] is obtained.

In 1999, Messaoudi [11] studied the following system

$$\begin{cases} u_t(x,t) = a(x)\phi(v(x,t))\, v_x(x,t) \\ v_t(x,t) = u_x(x,t) \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \quad x \in \mathbb{R}. \end{cases} \tag{3.2.8}$$

This system can be regarded as a model for a transverse motion of a nonhomogeneous vibrating string, where $a(x)$ is the density which is not constant. He established a blow up result for smooth data.

For more results of this nature, we refer the reader to [4], [12], [14] and [15].

3.3 Classical Theory of Heat Conduction

In the absence of deformation, heat propagation in one spatial dimension is governed by the following equation of balance of energy

$$e_t + q_x = 0 \tag{3.3.1}$$

where the internal energy e and the heat flux q are functions of (x, t) and subscript denotes a partial derivative with respect to the relevant variable. In Fourier's law of heat conduction, the internal energy depends on the absolute temperature only; i.e.

$$e = \hat{e}(\theta) \tag{3.3.2}$$

whereas the heat flux is given by the relation

$$q = -k(\theta)\,\theta_x. \tag{3.3.3}$$

Consequently, the evolution of heat flux and the absolute temperature is given by the system

$$\begin{aligned} q + K(\theta)\,\theta_x &= 0 \\ q_x + \hat{e}'\,(\theta)\theta_t &= 0 \end{aligned} \tag{3.3.4}$$

where k and $\hat{e}'$ are strictly positive functions characterizing the material in consideration. In the case where $\hat{e}'$ and k are independent of θ, we get the familiar linear heat equation

$$\theta_t = k\theta_{xx}, \quad k = \frac{K}{\hat{e}'}. \tag{3.3.5}$$

This equation provides a useful description of heat conduction under a large range of conditions and predicts and infinite speed of propagation; that is, any thermal disturbance at one point has an instantaneous effect elsewhere in the body. However this is not always the case. In fact, experiments showed that heat condition in some dielectric crystals at low temperatures is free of this paradox (infinite speed propagation) and disturbances which are almost entirely thermal, may propagate in finite speed. This phenomenon in dielectric crystals is called second sound (see [2], [3]).

These observations go back to 1948, when Cattaneo proposed, in place of (3.3.3), a new constitute relation

$$\tau(\theta)q_t + q = -k(\theta)\,\theta_x, \tag{3.3.6}$$

where τ and k are strictly positive functions depending on the absolute temperature. With this relation, the internal energy given by (3.3.2) is no longer compatible with the second law of thermodynamics. Coleman, Fabrizio and Owen [2] showed in 1982 that, if (3.3.6) is adopted then compatibility with thermodynamics requires that (2.3.2) should be replaced by

$$e = \tilde{e}(\theta, q) = a(\theta) + b(\theta)q^2, \tag{3.3.7}$$

where b is a function determined by τ and k. In particular $b(\theta) > 0$.

Thus (3.3.1), (3.3.6), and (3.3.7) combined together yield to the following system governing the evolution of θ and q

$$q_x - (a'(\theta) + b'(\theta)q^2)q_t + 2b(\theta)q \ \ q_t = 0 \tag{3.3.8}$$

$$\tau(\theta)q_t + q + k(\theta)\theta_x = 0. \tag{3.3.9}$$

Global existence and decay of classical solutions to the Cauchy problem, as well as to some initial boundary value problems have been established by Coleman, Hrusa, and Owen [3]. In their paper, the authors used a classical energy argument to prove their result. Concerning the formation of singularities, Messaoudi [9] studied the following system

$$\tau(\theta)q_t = q + k(\theta)\,\theta_x = 0 \tag{3.3.10}$$

$$c(\theta)\,\theta_t + q_x = 0$$

and showed, under the same restrictions on τ, c and k, that classical solutions to the Cauchy problem break down in finite time if the initial data are chosen small in L^∞ norm with large enough derivatives. This result has been generalized by Messaoudi [10] to the system (3.3.8) and (3.3.9).

Another approach to second sound is the one presented in [6], [19] and [20], where the authors introduce an internal parameter which accounts for the history memory effects of the heat flux. This approach gave rise to a new theory of heat conduction which we discuss in the next chapter.

Chapter 4

Non-Classical Theory of "Hyperbolic" Heat Conduction

4.1 Introduction

In this chapter, we investigate the breakdown of classical solutions for the following quasilinear system:

$$v_t - (h(v)p)_x = 0 \tag{4.1.1}$$

$$p_t - (\sigma(v))_x = f(v)p \tag{4.1.2}$$

This system describes the propagation of heat wave for rigid solids at very low temperature, below about 20^0K.

The first equation (4.1.1) comes from the balance of energy which, in the one-dimensional case, takes the form

$$(\epsilon(v))_t + q_x = 0, \tag{4.1.3}$$

where $v > 0$ is the absolute temperature, ϵ is the internal energy, and q is the one-dimensional heat flux. Equation (4.1.2) is the evolution equation for an internal parameter p, which is introduced to account for memory effects of the heat flux. The effect of memory may be considered, for example, as a functional of a history of temperature gradient,

$$q = -\alpha(v) \int_{-\infty}^{t} e^{-b(t-s)} v_x(x,s)ds, \quad \alpha(v) > 0, \quad b > 0 \tag{4.1.4}$$

by defining

$$p = \int_{-\infty}^{t} e^{b(t-s)} v_x(x,s)\, ds \tag{4.1.5}$$

Equation (4.1.4) can be equivalently replaced with

$$q = -\alpha(v)p, \tag{4.1.6}$$

$$p_t = -bp + v_x \tag{4.1.7}$$

Equation (4.1.7), related to (4.1.4) via (4.1.5) and (4.1.6), is however linear and does not fully describe the properties of heat propagation in solids. To improve the model one may generalize the history dependence of q by modifying equation (4.1.5), or, by introducing a suitable nonlinear dependence in (4.1.7)

$$p_t = g_1(v)\, v_x + g_2(v)p, \tag{4.1.8}$$

The functions α, g_1, and g_2 present in (4.1.6) and (4.1.8) are material functions. The second law of thermodynamics imposes the restrictions that $\alpha(v) = \psi_2\, v^2\, g_1(v)$ and $g_2(v) < 0$, where the constant $\psi_2 > 0$ comes from the Helmholtz free energy ψ which has the form $\psi = \psi_1(v) + \frac{1}{2}\,\psi_2\, v\, p^2$. We also make an assumption that $g_1(v) > 0$. Combining (4.1.3) with (4.1.8) we get the following system.

$$\epsilon(v)_t - (\alpha(v)p)_x = 0 \tag{4.1.9}$$

$$p_t + G_1(v)_x = g_2(v)p, \quad G_1'(v) = -g_1(v) \tag{4.1.10}$$

In the steady-state case, $p_t = 0$, equations (4.1.9), (4.1.10) lead to a nonlinear diffusion equation

$$c_v(v)v_t - (k(v)v_x)_x = 0, \tag{4.1.11}$$

where

$$k(v) = -\psi_2 \frac{v^2 g_1^2(v)}{g_2(v)} > 0$$

is the steady-state conductivity measured experimentally, $g = -k(v)v_x$ and $\epsilon'(v) = c_v(v)$ is the specific heat. If we set, $\epsilon(v) = v > 0$ with $\sigma = G_1 o\, \epsilon^{-1}$, $f = g_2 o\, \epsilon^{-1}$, $h = \alpha\, o \in^{-1}$, $\sigma'(v) > 0$, $h(v) > 0$ and $f(v) < 0$, then we obtain from (4.1.9) and (4.1.10) the following system,

$$\begin{cases} v_t - p_x = 0 \\ p_t - (\sigma(v))_x = f(v)p \\ v(x,0) = v_0(x), \quad p(x,0) = p_0(x) \end{cases} \tag{4.1.12}$$

Now, before the investigation of the break-down of the classical solution of the system (4.1.12), we introduce the following concepts, which will be useful in the proof of our main result.

Mean value Theorem [21]

Let S be an open subset of R^n and assume that $f : S \to R^m$ is differentiable at each point of S. Let x and y be two points in S such that the line joining x and y $L_1(x, y) \subseteq S$. Then there is a point z on $L_1(x, y)$ such that

$$f(Y) - f(X) = (Y - X) \cdot \nabla f(Z) \tag{4.1.13}$$

Z is given by

$$Z = t_0(Y - X) + X \quad 0 < t_0 < 1.$$

If we take $n = 2$ and $Y = (x, y)$ and $Y = (x + h,\, y + k)$, (4.1.3) reduces to

$$f(x + h,\, y + k) - f(x, y) = hf_x(x + \theta_1 h,\, y + \theta_2 k)$$

$$+k\, f_y(x + \theta_1 h,\, y + \theta_2 k), \quad 0 < \theta_1, \theta_2 < 1.$$

Young's Inequality [8]

Let a, b be nonnegative, $1 < p < \infty$, $\quad \frac{1}{p} + \frac{1}{q} = 1$. Then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

If we takes $a = \sqrt{d}\, F$, $\quad b = \frac{M}{\sqrt{d}}$ and $p = 2$. We get Cauchy-Schwarz inequality

$$MF \leq \frac{d}{2}\, F^2 + \frac{1}{2d}\, M^2.$$

for all $d > 0$.

4.2 Formation of Singularities

This section is devoted to the statement and the proof of our main result. We first begin with a lemma which gives a pointwise upper bound on the solution in terms of the initial data.

Lemma 4.2.1 Assume that $\sigma' > 0$ and f are C^1 functions, with $|f(y)| \leq C$, $\forall y \in \mathbb{R}$ and let $v_0, p_0 \in H^2(\mathbb{R})$ be given then any solution

(v,p) to the problem

$$\begin{cases} v_t - p_x = 0 \\ p_t - (\sigma(v))_x = f(v)p \\ v(x,0) = v_0(x), \quad p(x,0) = p_0(x), \quad x \in \mathbb{R}, \quad t \geq 0 \end{cases} \tag{4.2.1}$$

satisfies

$$\max_{\substack{(x,t)\\ 0\leq t\leq T}} \{|v(x,t)| + |p(x,t)|\} \leq k' \max_x \{|v_0(x)| + |p_0(x)|\}, \tag{4.2.2}$$

where k' is a positive constant independent of v and p.

Proof. First, we write the system (4.2.1) as $\dot{x} = Ax + B$, that is

$$\begin{bmatrix} v \\ p \end{bmatrix}_t = \begin{bmatrix} 0 & 1 \\ \sigma'(v) & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix}_x + \begin{bmatrix} 0 \\ f(v) \end{bmatrix} p.$$

By putting

$$A = \begin{pmatrix} 0 & 1 \\ \sigma'(v) & 0 \end{pmatrix}, \text{ we obtain } \quad |A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ +\sigma'(v) & -\lambda \end{vmatrix} = \lambda^2 - \sigma'(v) = 0$$

whose roots are:

$$\lambda_1 = \sqrt{\sigma'(v)}, \qquad \lambda_2 = -\sqrt{\sigma'(v)}.$$

This shows that the system is strictly hyperbolic with two real distinct eigenvalues.

Now for $\lambda_1 = \sqrt{\sigma'(v)}$, the corresponding eigenvector:

$$\begin{pmatrix} -\sqrt{\sigma'(v)} & 1 \\ \sigma'(v) & -\sqrt{\sigma'(v)} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -\sqrt{\sigma'(v)}\, x_1 + y_1 = 0 \\ \sigma'(v) x_1 - \sqrt{\sigma'(v)}\, y_1 = 0 \end{cases}$$

$$\Rightarrow \quad e = \begin{pmatrix} 1 \\ \sqrt{\sigma'(v)} \end{pmatrix} \text{ is an eigenvector; so } e^{\perp} = \begin{pmatrix} r_v \\ r_p \end{pmatrix} = \begin{pmatrix} -\sqrt{\sigma'(v)} \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad r_v = -\sqrt{\sigma'(v)} \text{ and } r_p = 1$$

$r = -\int_0^v \sqrt{\sigma'(\xi)}\, d(\xi) + \phi(p) \Rightarrow r_p(x,t) = \phi'(p) = 1 \Rightarrow \qquad \phi(p) = p(x,t)$

So $\quad r(x,t) = p(x,t) - \int_0^{v(x,t)} \sqrt{\sigma'(\xi)}\ d\xi$ (4.2.3)

which can be written as

$$r(x,t) = p(x,t) - A(v), \tag{4.2.4}$$

where

$$A(v) = \int_0^v \sqrt{\sigma'(\xi)}\ d\xi. \tag{4.2.5}$$

Similarly we can obtain,

$$s(x,t) = p(x,t) + A(v). \tag{4.2.6}$$

Now we introduce the differential operators

$$\partial_t^+ = \frac{\partial}{\partial t} + \sqrt{\sigma'(v)}\,\frac{\partial}{\partial x} \tag{7.2.7a}$$

and

$$\partial_t^- = \frac{\partial}{\partial t} - \sqrt{\sigma'(v)}\,\frac{\partial}{\partial x}, \tag{7.2.7b}$$

and compute

$$\begin{aligned} \partial_t^+ r &= r_t + \sqrt{\sigma'(v)}\, r_x \qquad (4.2.8)\\ &= \left[-\sqrt{\sigma'(v)}\, v_t + p_t\right] + \sqrt{\sigma'(v)}\left[-\sqrt{\sigma'(v)}\, v_x + p_x\right]\\ &= \left[-\sqrt{\sigma'(v)}\, v_t + \sqrt{\sigma'(v)}\, p_x\right] + [p_t - \sigma'(v)\, v_x]\\ &= \sqrt{\sigma'(v)}\,[-v_t + p_x] + [p_t - \sigma'(v)\, v_x] \end{aligned}$$

$$(4.2.9)$$

From (4.2.1), $p_x - v_t = 0$ and it can be submitted in (4.2.9) to obtain

$$\partial_t^+ r = [p_t - \sigma'(v)\, v_x]\,. \qquad (4.2.10)$$

Also, from (4.2.1) we have

$$p_t = (\sigma'(v))_x + f(v)p. \qquad (4.2.11)$$

Substituting (4.2.11) in (4.2.10) we arrive at

$$\partial_t^+ r = +(\sigma'(v))_x + f(v)p - \sigma'(v)\, v_x \qquad (4.2.12)$$

$$\partial_t^+ r = f(v)p. \qquad (4.2.13)$$

Similar computations also yield

$$\begin{aligned} \partial_t^- s &= \left[\sqrt{\sigma'(v)} v_t + p_t\right] - \left(\sqrt{\sigma'(v)}\right)\left[\sqrt{\sigma'(v)}\, v_x + p_x\right]\\ &= \sqrt{\sigma'(v)}\,[v_t - p_x] + [p_t - \sigma'(v)\, v_x]\\ \partial_t^- s &= p_t - \sigma(v)\, v_x = f(v)p. \end{aligned}$$

$$(4.2.14)$$

We then define

$$R(t) := \max_x |r(x,t)|$$

$$S(t) := \max_x |s(x,t)| \tag{4.2.15}$$

The maxima in (4.2.15) are attained because R and S decay at infinity, [R and S are $H^1(\mathcal{R})$ and $\lim_{x\to\pm\infty} u(x) = 0$ if $u \,\epsilon\, H^1(\mathcal{R})$].

For any $t \in (0,T)$, we can choose x_1 and x_2 so that

$$R(t) = |r(x_1,t)| \text{ and } S(t) = |s(x_2,t)|. \qquad (4.2.16).$$

Therefore, for any $h \in (0,t)$, we have

$$R(t-h) \geq \left| r\left(x_1 - h\sqrt{\sigma'(v)}, t-h\right)\right| \tag{4.2.17}$$

since $\quad R(t-h) = \max_x |r(x_1, t-h)|$.

Now

$$\begin{aligned} R(t) - R(t-h) &\leq |r(x_1,t)| - |r(x_1 - h\sqrt{\sigma'(v)},\, t-h)| \\ &\leq |r(x_1,t) - r(x_1 - h\sqrt{\sigma'(v)},\, t-h)| \end{aligned} \tag{4.2.18}$$

$$R'(t) = \lim_{h\to 0} \frac{R(t) - R(t-h)}{h} \tag{4.2.19}$$

for almost every $t \in (0,T)$.

$$R'(t) \;\leq\; \lim_{h\to 0}\left|\frac{-h\sqrt{\sigma'(v)}r_x\left(x_1-h\alpha\sqrt{\sigma'(v)},t-\alpha h\right)-h\,r_t\left(x_1-h\alpha\sqrt{\sigma'(v)},\,t-\alpha h\right)}{h}\right|$$

$$R'(t) \;\leq\; \lim_{h\to 0}\left|\sqrt{\sigma'(v)}\,r_x\left(x_1-h\sqrt{\sigma'(v)}\,\alpha,\,t-h\right)-r_t\left(x_1-h\sqrt{\sigma'(v)}\alpha,\,t-h\right)\right|$$

$$0<\alpha<1$$

$$R'(t) \leq \left|\sqrt{\sigma'(v)}\,r_x(x_1,t)-r_t(x_1,t)\right|. \tag{4.2.20}$$

Substitute (4.2.13) into (4.2.20) to obtain

$$R'(t) \leq \left|\partial_t^+ r\right| \leq |f(v)\,p| \tag{4.2.21}$$

Similarly, we can obtain

$$S'(t) \leq \left|\partial_t^- s\right| \leq |f(v)\,p| \tag{4.2.22}$$

Now we add (4.2.21) to (4.2.22)

$$\frac{d}{dt}[R(t)+S(t)] \leq 2|f(v)\rho|. \tag{4.2.23}$$

Since $|f(v)| \leq c$ and from (4.2.4) and (4.2.5) we compute

$$p(x,t)=\frac{r(x,t)+s(x,t)}{2} \tag{4.2.24}$$

Substituting (4.2.24) in (4.2.23), we obtain

$$\frac{d}{dt}\,[R(t)+S(t)] \leq c\,[R(t)+S(t)] \tag{4.2.25}$$

for almost every $t \in [0,T]$.

Integration of both sides of (4.2.25) gives

$$R(t) + S(t) \leq (R(0) + S(0)) + c \int_0^t (R(\eta) + s(\eta))\, d\eta \tag{4.2.26}$$

By using Gronwall's inequality, we obtain

$$(R(t) + S(t)) \leq (R(0) + S(0))\, e^{ct}, \;\; \forall\, t \in [0, T]. \tag{4.2.27}$$

Therefore, (4.2.2) follows.

Theorem 4.2.2 Let σ' and f as in the lemma (4.2.1). Assume further that $\sigma'' > 0$.

Then for any $L > 0$, there exists initial data $v_0, p_0 \in H^2(\mathbb{R})$ for which the solution (v, p) blows up in finite time $T < L$.

Proof. We take an x-partial derivative of (4.2.13) to get

$$\begin{aligned}
\left(\partial_t^+ r\right)_x &= \left(r_t + \sqrt{\sigma'(v)}\, r_x\right)_x && (4.2.28)\\
&= r_{tx} + \frac{1}{2}\,\sigma''(v)\,\sigma'^{-\frac{1}{2}}(v)\, v_x\, r_x + \sqrt{\sigma'(v)}\, r_{xx} \\
&= \left(r_{tx} + \sqrt{\sigma'(v)}\, r_{xx}\right) + \frac{1}{2}\,\sigma''(v)\,\sigma'^{-\frac{1}{2}}\, v_x\, r_x && (4.2.29)\\
&= \partial_t^+ r_x + \frac{1}{2}\,\sigma''(v)\,\sigma'^{-\frac{1}{2}}(v)\, v_x\, r_x = (f(v)p)_x && (4.2.30)
\end{aligned}$$

From (4.2.4) and (4.2.5), we have

$$v_x = \frac{s_x - r_x}{2\sqrt{\sigma'(v)}}, \; p_x = \frac{s_x + r_x}{2}. \tag{4.2.31}$$

Substituting (4.2.31) in (4.2.30) we obtain

$$\partial_t^+ r_x + \frac{1}{2}\,\sigma''(v)\,v_x\,\sigma'^{-\frac{1}{2}}(v)\,r_x = f'(v)\,v_x\,p + f(v)\,p_x \tag{4.2.32}$$

that is

$$\partial_t^+ r_x = \frac{-1}{2}\,\sigma''(v)\frac{s_x - r_x}{2\sqrt{\sigma'(v)}}\,\sigma'^{-\frac{1}{2}}(v)\,r_x + f'(v)\left(\frac{s_x - r_x}{2\sqrt{\sigma'(v)}}\right)\left(\frac{r+s}{2}\right) + f(v)\left(\frac{r_x + s_x}{2}\right)$$

Now subtracting (4.2.4) from (4.2.6)

$$\begin{aligned}
r - s &= -2\int_0^v \sqrt{\sigma'(\xi)}\,d\xi \\
\partial_t^+(r-s) &= \left[-2\int_0^v \sqrt{\sigma'(\xi)}\,d\xi\right]_t + \sqrt{\sigma'(v)}\left[-2\int_0^v \sqrt{\sigma'(\xi)}\,d\xi\right]_x \\
&= -2\sqrt{\sigma'(v)}\,v_t + \sqrt{\sigma'(v)}\left(-2\sqrt{\sigma'(v)}\right)\,v_x \\
&= -2\sqrt{\sigma'(v)}\left[p_x + \sqrt{\sigma'(v)}\,v_x\right] \\
&= -2\sqrt{\sigma'(v)}s_x
\end{aligned}$$

$$\Rightarrow \quad s_x = \frac{-1}{2\sqrt{\sigma'(v)}}\,\partial_t^+\,(r-s). \tag{4.2.33}$$

Now let

$$w = \alpha(v) r_x \tag{4.2.34}$$

$$\begin{aligned}
\partial_t^+(\alpha(v)r_x) &= \left[\partial_t^+\alpha(v)\right] r_x + \alpha(v)\partial_t^+ r_x \\
&= \left[\alpha_t(v) + \sqrt{\sigma'(v)}\,\alpha_x(v)\right] r_x + \alpha(v)\left[\frac{-1}{2}\,\sigma''(v)\,\sigma'^{-\frac{1}{2}}(v)\left(\frac{s_x - r_x}{2\sqrt{\sigma'(v)}}\right) r_x + \right. \\
& \left. f'(v)\left(\frac{s_x - r_x}{2\sqrt{\sigma'(v)}}\right)\left(\frac{r+s}{2}\right) + f(v)\left(\frac{r_x + s_x}{2}\right)\right] \\
&= \alpha'(v)\left[v_t + \sqrt{\sigma'(v)}\,v_x\right] r_x + \alpha(v)\left(\left[\frac{-\sigma''(v)(s_x - r_x)}{4\sigma'(v)}\,r_x\right]\right. \\
&+ \left.\frac{f'(v)(s_r - r_x)}{4\sqrt{\sigma'(v)}}\,(r+s) + f(v)\left(\frac{r_x + s_x}{2}\right)\right). \qquad (4.2.35)
\end{aligned}$$

At this point we choose $\alpha(v)$ so that

$$\alpha'(v)\left[v_t + \sqrt{\sigma'(v)}\,v_x\right] r_x - \frac{\sigma''(v)\,s_x r_x\,\alpha(v)}{4\sigma'(v)} = 0$$

By using (4.1.12), we arrive at

$$\alpha'(v)\left[p_x + \sqrt{\sigma'(v)}\,v_x\right]\,r_x - \frac{\sigma''(v)s_x\,r_x\,\alpha(v)}{4\alpha'(v)} = 0$$

and exploiting (4.2.6) we get

$$\alpha'(v)\left[p_x + \sqrt{\sigma'(v)}\,v_x\right] r_x = \frac{\sigma''(v)\alpha(v)r_x[p_x + \sqrt{\sigma'(v)}\,v_x]}{4\sigma'(v)}. \qquad (4.2.36)$$

Therefore α satisfies

$$\alpha'(v) - \frac{\sigma''(v)\alpha(v)}{4\sigma'(v)} = 0. \qquad (4.2.37)$$

Solving (4.2.37) gives

$$\alpha(v) = [\sigma'(v)]^{\frac{1}{4}}. \qquad (3.2.38)$$

By substituting (4.2.38) in (4.2.35) we obtain

$$\partial_t^+ w = \sigma'(v)^{\frac{1}{4}} \left[\frac{\sigma''(v)}{4\sigma'(v)} r_x^2 + \frac{f'(v)(s_x - r_x)}{4\sqrt{\sigma'(v)}} (r+s) + f(v) \left(\frac{r_x + s_x}{2} \right) \right] \tag{4.2.39}$$

$$\partial_t^+ w = [\sigma'(v)]^{\frac{1}{4}} \left[\frac{\sigma''(v)}{4\sigma'(v)} r_x^2 - \left(\frac{f'(v)}{4\sqrt{\sigma'(v)}} (r+s) - \frac{f(v)}{2} \right) r_x + \left(\frac{f'(v)(r+s)}{4\sqrt{\sigma'(v)}} + \frac{f}{2} \right) s_x \right] \tag{4.2.40}$$

$$\partial_t^+ w = \frac{\sigma''(v)}{4\sigma'(v)} \frac{(\sigma(v) r_x)^2}{(\sigma'(v))^{\frac{1}{4}}} - \left(\frac{f'(v)}{4\sqrt{\sigma'(v)}} (r+s) - \frac{f(v)}{2} \right) \left(\sigma'(v)^{\frac{1}{4}} r_x \right) \sigma'(v)^{\frac{1}{4}} \left[\frac{f'(v)(r+s)}{\sqrt{\sigma'(v)}} + \frac{f}{2} \right] s_x \tag{4.2.41}$$

$$= \frac{\sigma''(v)}{4\sigma'(v)^{\frac{5}{4}}} w^2 - \left(\frac{f'(v)}{4\sqrt{\sigma'(v)}} (r+s) - \frac{f(v)}{2} \right) w + \sigma'(v)^{\frac{1}{4}} \left[\frac{f'}{4\sqrt{\sigma'(v)}} (r+s) + \frac{f}{2} \right] s_x. \tag{4.2.42}$$

The last terms in (4.2.42) can be handled as follows:

$$H = [\sigma'(v)]^{\frac{1}{4}} \left[\frac{f'}{4\sqrt{\sigma'(v)}} (r+s) \right] s_x = \sigma'(v)^{\frac{1}{4}} \left[\frac{f'}{4\sqrt{\sigma'(v)}} (2r + 2A(v)) \right] s_x. \tag{4.2.43}$$

Substituting the value of s_x from (4.2.33) into (4.2.43) to arrive at

$$\begin{aligned} H &= \sigma'^{\frac{1}{4}}(v)\left[\frac{f'}{4\sqrt{\sigma'(v)}}(2r+2A(v))\right]\left(\frac{-\partial_t^+(r-s)}{2\sqrt{\sigma'(v)}}\right) \\ &= \frac{-1}{4}(\sigma'(v))^{\frac{-3}{4}}[f'(v)(r+A(v)]\partial_t^+(r-s) \\ &= \frac{-1}{4}(\sigma'(v))^{\frac{-3}{4}}f'(v)r\,\partial_t^+(r-s)-\frac{\sigma'(v)^{\frac{-3}{4}}}{4}f'(v)A(v)\partial_t^+(r-s). \end{aligned} \tag{4.2.44}$$

Straightforward calculations then give

$$\partial_t^+(r-s)=\partial_t^+(-2A(v))=\partial_t^+\left(-2\int_0^v\sqrt{\sigma'(\xi)}\,d\xi\right)=-2\sqrt{\sigma'(v)}\ \ \partial_t^+\,v$$

Now (4.2.44) gives

$$H=\frac{1}{2}(\sigma'(v))^{\frac{-1}{4}}f'(v)r\,\partial_t^+v+\frac{1}{2}\sigma'(v)^{-\frac{1}{4}}f'(v)A(v)\partial_t^+v \tag{4.2.45}$$

The right side of (4.2.45) can be written as

$$\frac{1}{2}(\sigma'(v))^{\frac{-1}{4}}f'(v)r\,\partial_t^+v+\frac{1}{2}\sigma'(v)^{\frac{-1}{4}}f'(v)A(v)\partial_t^+\,v=\frac{1}{2}\left[\partial_t^+\left(r\int_0^v\sigma(\xi)^{\frac{-1}{4}}f'(\xi)d\xi\right)\right.$$
$$\left.-\left(\partial_t^+r\right)\int_0^v\sigma'(\xi)^{\frac{-1}{4}}\,f'(\xi y)\,d\xi\right]+\partial_t^+\left[\frac{1}{2}\int_0^v\sigma'(\xi)^{\frac{-1}{4}}f'(\xi)A(\xi)\,d\xi\right]. \tag{4.2.46}$$

Now substitute (4.2.46) in (4.2.42)

$$\begin{aligned}
\partial_t^+ w &= \frac{\sigma'' w^2}{4\sigma'^{\frac{5}{4}}} - \left(\frac{f'}{4\sqrt{\sigma'}}(r+s) - \frac{f}{2}\right) w + \partial_t^+ \int_0^v f(\xi)\sigma'(\xi)^{\frac{3}{4}}\, d\xi + \\
&\frac{1}{2}\left[\partial_t^+ r \int_0^v \sigma'^{\frac{-1}{4}} f'(\xi)\, d\xi - \left(f(v)\left(\frac{r+s}{2}\right)\int_0^v \sigma'^{\frac{-1}{4}}(\xi) f'(\xi)\, d\xi\right)\right] + \\
&\partial_t^+ \left[\frac{1}{2}\int_0^v \sigma'^{\frac{-1}{4}} f'(\xi)\, A(\xi) d\xi\right]. \qquad (4.2.47)
\end{aligned}$$

$$\begin{aligned}
\partial_t^+ w &= \frac{\sigma''(v) w^2}{4(\sigma'(v))^{\frac{5}{4}}} - \left(f(v)\left(\frac{r+s}{4}\right)\int_0^v \sigma'(\xi)^{\frac{-1}{4}}\, f'(\xi)\, d\xi\right) \\
&- \left(\frac{f'(v)}{4\sqrt{\sigma'(v)}}(r+s) - \frac{f}{2}\right) w \\
&+ \partial_t^+ \left[\frac{r}{2}\int_0^v \sigma'(\xi)^{\frac{-1}{4}} f'(\xi)\, dy - \int_0^v \left[f(\xi)\sigma'^{\frac{3}{4}}(\xi) - \frac{1}{2}\sigma'(\xi)^{\frac{-1}{4}} f'(\xi) A(\xi)\right] d\xi\right] \qquad (4.2.48) \\
&= \frac{\sigma''(v) w^2}{4\sigma'^{\frac{5}{4}}} - \left(\frac{f'}{4\sqrt{\sigma'}}(r+s) - \frac{f}{2}\right) w \\
&- f(v)\left(\frac{r+s}{2}\right)\int_0^v \sigma'(\xi)^{-\frac{1}{4}} f'(\xi)\, d\xi + \partial_t^+ g \qquad (4.2.49)
\end{aligned}$$

where

$$g = \frac{r}{2}\int_0^v \sigma'^{\frac{-1}{4}}(\xi) f'(\xi)\, d\xi - \int_0^v \left[f\sigma'^{\frac{3}{4}} - \frac{1}{2}\sigma'^{\frac{-1}{4}}\, f' A\right](\xi)\, d\xi \qquad (4.2.50)$$

$$w - g = k, \qquad w = k + g$$

$$\partial_t^+ k = \frac{\sigma''(v)(k+g)^2}{4\sigma'^{\frac{5}{4}}(v)} - \left(\frac{f'(v)}{4\sqrt{\sigma'}}(r+s) - \frac{f}{2}\right)(k+g) - \frac{f(v)(r+s)}{2} \int_0^v \left(\sigma'^{\frac{-1}{4}} f'\right)(\xi) d\xi \tag{4.2.51}$$

$$= \frac{\sigma''(v)}{4\sigma'^{\frac{5}{4}}(v)}\left[g^2 + 2kg + k^2\right] - \left(\frac{f'(v)(r+s)}{4\sqrt{\sigma'(v)}} - \frac{f}{2}\right)(k+g) - \frac{f(v)(r+s)}{2}\int_0^v (\sigma'^{\frac{-1}{4}} f')(\xi) d\xi \tag{4.2.52}$$

$$= \frac{\sigma''(v)}{4\sigma'(v)^{\frac{5}{4}}}k^2 + \left(\frac{\sigma''(v)}{2\sigma'^{\frac{5}{4}}}\, g - \frac{f'(v)(r+s)}{2\sqrt{\sigma'(v)}} - \frac{f}{2}\right) k + \left[\frac{\sigma''(v)}{4\sigma'^{\frac{5}{4}}} g^2 - \left(\frac{f'(v)}{4\sqrt{\sigma'}}(r+s) - \frac{f}{2}\right) g - \frac{f(v)(r+s)}{2}\int_0^v (\sigma'^{\frac{-1}{4}} f')(\xi)\, d\xi\right]. \tag{4.2.53}$$

By choosing initial data small enough (in L^∞ norm), we are guaranteed to have

$$a := inf\left(\frac{\sigma''(v)\,\sigma'^{\frac{-5}{4}}(v)}{4}\right) > 0. \tag{4.2.54}$$

We set ,

$$m := max\left|\frac{\sigma''(v)}{4\sigma'^{\frac{5}{4}}(v)}\, g^2 - \left(\frac{f'(v)}{4\sqrt{\sigma'(v)}}(r+s) - \frac{f}{2}\right) g - \frac{f(v)(r+s)}{2}\int_0^v (\sigma'^{\frac{-1}{4}} f')(\xi) d\xi\right| \tag{4.2.55}$$

$$M = \left[\frac{\sigma''(v)}{2\sigma'^{\frac{5}{4}(v)}}\, g - \frac{f'(v)(r+s)}{2\sqrt{\sigma'(v)}} - \frac{f}{2}\right] \tag{4.2.56}$$

Thus, we have

$$\partial_t^+ k \geq ak^2 + Mk - m \tag{4.2.57}$$

From Young's inequality we can get the following

$$Mk \leq \frac{\delta}{2}\,k^2 + \frac{1}{2\delta}\,M^2 \qquad \forall \delta > 0 \tag{4.2.58}$$

So (4.2.57) becomes

$$\partial_t^+\,k \geq ak^2 - \frac{\delta}{2}\,k^2 - \frac{1}{2\delta}\,M^2 - m \tag{4.2.59}$$

Taking $\delta = a$, we arrive at

$$\partial_t^+ k \geq \frac{a}{2}\,k^2 - \frac{1}{2a}\,M^2 - m \tag{4.2.60}$$

$$\partial_t^+ k \geq \frac{a}{2}\,k^2 - \frac{a}{2}\,B^2 \tag{4.2.61}$$

where $B^2 := \dfrac{M^2}{2a} + \dfrac{2m}{a}$

$$\partial_t^+\,k \geq \frac{a}{2}\left(k^2 - B^2\right). \tag{4.2.62}$$

It suffices to choose v_0 and p_o small enough in L^∞ norm with derivatives large enough, such that near the zero we get:

$$\frac{1}{2}\,k^2(0) > B^2 \;\text{ and }\; \frac{4}{gk(0)} < L.$$

Now, (4.2.62) reduces to:

$$\partial_t^+ k(0) \geq \frac{a}{4}k^2(0) \tag{4.2.63}$$

hence

$$\frac{dk(t)}{dt} \geq \frac{a}{4}k^2(t), \tag{4.2.64}$$

$$\Rightarrow \frac{dk(t)}{k^2(t)} \geq \frac{a}{4}dt.$$

Integration along the forward characteristics then yields

$$
\begin{aligned}
\frac{-1}{k(t)} \quad + \quad \frac{1}{k(0)} &\geq \frac{a}{4}t \\
\frac{1}{k(t)} &\leq \frac{1}{k(0)} - \frac{a}{4}t \\
k(t) &\geq \frac{1}{\frac{1}{k(0)} - \frac{a}{4}t}
\end{aligned}
\tag{4.2.65}
$$

Therefore, $k(t) \longrightarrow \infty$, in a time $t^* \leq \frac{4}{ak(0)} < L$.

Therefore (4.2.65) shows that $k(t)$ (hence r_x) blows up in finite time.

Remark 4.2.1 The blow up of r_x implies that either p_x or v_x (hence v_t or p_t) blows up in finite time, however the solution (p, v) remains bounded in the L^∞ norm.

Remark 4.2.2 A similar result can also be obtained for certain initial boundary value problem.

Remark 4.2.3 The same result can be proved if we only assume that f is C^1 near zero.

References

1. Bernard, K., Introductory Linear Algebra, Sixth Edition, Printice Hall, 1997.

2. Coleman, B.D., M. Fabrizio, and D.R. Owen, On the thermodynamics of second sound of dielectric crystals, *Arch. Rational Mech. Analysis* **80**(1982), 135-158.

3. Coleman, B.D., W.J. Hrusa, and D. Owen, Stability of equilibrium for a nonlinear hyperbolic system describing heat propagation by second sound in solids, *Arch. Rational Mech. Anal.* **94** (1986), 267-289.

4. Kirane M. and S. Messaoudi, Breakdown in finite time of solutions to a one-dimensional wave equation, *Revista Mathematica Complutense* vol. **XIII 2** (2000), 413-422.

5. Lax P.D., Development of singularities in solutions of nonlinear hyperbolic partial differential equations, *J. Math. Physics* **5** (1964), 611-613.

6. Li H. and K. Saxton, Asymptotic behavior of solutions to quasilinear hyperblic equations with nonlinear damping, *Quart. Appl. Math.* **61**, no.2, (2003), 295-313.

7. MacCamy R.C. and V.J. Mizel, Existence and nonexistence in the large solutions of quasilinear wave equations, *Arch. Rational Mech. Anal.* **25** (1967), 299-320.

8. McOwen R., Partial Differential Equations Methods & Applications, First Edition, *Frentice Hall* 1995.

9. Messaoudi S.A., Formation of singularities in heat propagation guided by second sound, *J. Diff. Eqns.* **130** (1996), 92-99.

10. Messaoudi S.A., On the existence and nonexistence of solutions of a nonlinear hyperbolic system describing heat propagation by second sound, *Applicable Analysis* **73** (1999), 485-496.

11. Messaoudi S.A., Formation of singularities in solutions of a wave equation, *Applied Math. Letters* **12** (1999), 23-28.

12. Messaoudi S.A., Blow up in solutions of an equation describing a transverse motion of a nonhomogeneous string, *Arab J. Math. Sc.* vol. **6(1)** (2000), 75-84.

13. Messaoudi S.A., Gradient Catastrophe in the classical solutions of nonlinear hyperbolic systems, *Journal of Partial Differential equations* vol. **13** (2000). 28-35.

14. Messaoudi S.A., Formation of singularities in solutions of a quasilinear strictly hyperbolic system, *Dynamics of Continuous, Discrete, and Impulsive Systems* (to appear).

15. Messaoudi, S.A., Global existence, exponential decay, and Blow up in one-dimensional quasi-linear hyperbolic systems: A unified Approach, *Dynamics of Continuous Discrete, and Impulsive Systems* (to appear).

16. Mosinsky W., Gradient catastrople of nonconservative hyperbolic systems, *J. Math. Anal.* **61** (1977), 672-688.

17. O'Neil P., Beginning Partial Differential Equations, *John Wiely and Sons,* First Edition 1999.

18. Royden, H., Real Analysis, *3rd edition,* Prentice Hall, 1988.

19. Saxton K., R. Saxton, and W. Kosinski, On second sound at the critical temperature, *Quarterly of Applied Math.* vol. **57 (4)**, (1999) 723-740.

20. Saxton, K. and R. Saxton, Nonlinearity and memory effects in low temperature heat propagation, *Arch. Rat. Mech.* **52** (2000), 127-142.

21. Tom M. Apostol, Mathematical Analysis, *3rd edition,* Narsa, Publishing House 1997.

22. Zachmanoglou E. and D. Thoe, Introduction to Partial Differental Equations with applications, 1976.